

# The thermal conductivity of the spin-1/2 $XXZ$ chain at arbitrary temperature

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December 24, 2001

## Abstract

Motivated by recent investigations of transport properties of strongly correlated 1d models and thermal conductivity measurements of quasi 1d magnetic systems we present results for the integrable spin-1/2  $XXZ$  chain. The thermal conductivity  $\kappa(\omega)$  of this model has  $\text{Re } \kappa(\omega) = \tilde{\kappa} \delta(\omega)$ , i.e. it is infinite for zero frequency  $\omega$ . The weight  $\tilde{\kappa}$  of the delta peak is calculated exactly by a lattice path integral formulation. Numerical results for wide ranges of temperature and anisotropy are presented. The low and high temperature limits are studied analytically.

*PACS:* 74.25.Fy; 75.10.Jm; 05.50.+q

*Keywords:* Transport properties; Thermal conductivity; Integrable system;  $XXZ$  model; Kubo formula; Quantum transfer matrix

## 1 Introduction

Transport properties of strongly correlated quantum systems have recently attracted strong theoretical and experimental interest. For the  $XXZ$  chain as a quantum spin model (or alternatively as a lattice gas model) the spin transport (electrical transport) was investigated analytically [1] by use of a method suggested in [2] and numerically [3, 4] with so far inconclusive results about the finite temperature Drude weight. In [5–7] the thermal conductivity of the quasi one-dimensional magnetic system  $(\text{Sr,Ca})_{14}\text{Cu}_{24}\text{O}_{41}$  was investigated showing anomalous transport properties along the chain directions. In [7] it was argued that the large thermal conductivity can not be explained in terms of phonons.

Here we want to use an approach as microscopic as possible and apply Kubo's theory to the strongly correlated spin-1/2  $XXZ$  chain. This approach closely follows [8]. We explicitly avoid the notion of particles and Boltzmann equations that take account of scattering

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as is commonly done in Fermi liquid theory. In strongly correlated low-dimensional quantum systems there is neither a particle picture with finite quasi-particle weight nor do we restrict ourselves to low temperatures in comparison to some reference (“Fermi”) energy. Rather on the opposite, we are interested in temperatures comparable to the exchange energy of neighboring spins.

The Kubo formulas [9, 10] are obtained within linear response theory and yield the (thermal) conductivity  $\kappa$  relating the (thermal) current  $\mathcal{J}_E$  to the (temperature) gradient  $\nabla T$

$$\mathcal{J}_E = \kappa \nabla T, \quad (1.1)$$

where

$$\kappa(\omega) = \beta \int_0^\infty dt e^{-i\omega t} \int_0^\beta d\tau \langle \mathcal{J}_E(-t - i\tau) \mathcal{J}_E \rangle, \quad (1.2)$$

and  $\beta$  is the reciprocal temperature  $1/(k_B T)^1$ . The current-current correlation function is to be evaluated in thermal equilibrium and poses the main problem of our work. However, as already pointed out in [8, 11–15] the total thermal current operator  $\mathcal{J}_E$  commutes with the Hamiltonian  $\mathcal{H}$  of the  $XXZ$  chain. Hence we find

$$\kappa(\omega) = \frac{1}{i(\omega - i\epsilon)} \beta^2 \langle \mathcal{J}_E^2 \rangle, \quad (\epsilon \rightarrow 0+), \quad (1.3)$$

with  $\text{Re } \kappa(\omega) = \tilde{\kappa} \delta(\omega)$  where

$$\tilde{\kappa} = \pi \beta^2 \langle \mathcal{J}_E^2 \rangle. \quad (1.4)$$

As a consequence, the thermal conductivity at zero frequency is infinite! The property of a non-decaying thermal current is a feature of 1d systems in particular of integrable systems with many integrals of motion (for a discussion see [23]). We expect that additional interactions, especially residual interchain couplings in quasi one-dimensional materials will lead to a broadening of the delta peak, however with same weight. Instead of speculations about the possible scenarios of such a broadening we want to present results for the weight  $\tilde{\kappa}$  as a function of temperature and anisotropy.

The paper is organized in the following way. In Sect.2 we discuss the integrability of the  $XXZ$  chain and identify the thermal current as one of the integrals of motion. In Sect.3 we present our method of calculation of thermodynamical quantities and discuss the results. In Sect.4 we treat analytically the low and high temperature limits and give a summary in Sect.5.

## 2 Thermal current and conserved quantities

Here we begin the microscopic approach to the heat conductivity of the spin-1/2 quantum chain with a construction of the conserved currents. The Hamiltonian of the  $XXZ$  model

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<sup>1</sup>In this paper we set  $k_B = 1$ .

on a periodic lattice of size  $L$  is

$$\mathcal{H} = \sum_{k=1}^L h_{kk+1},$$

$$h_{kk+1} = J \left( \sigma_k^+ \sigma_{k+1}^- + \sigma_{k+1}^+ \sigma_k^- + \frac{\Delta}{2} \sigma_k^z \sigma_{k+1}^z \right), \quad (2.1)$$

where  $\sigma_k^\pm = (\sigma_k^x \pm i\sigma_k^y)/2$  and  $\sigma_k^x, \sigma_k^y, \sigma_k^z$  denote the usual Pauli matrices acting on the  $k$ th space.

In this paper we restrict ourselves to the critical regime  $-1 < \Delta \leq 1$  where the system displays algebraically decaying correlation functions at zero temperature. The anisotropy parameter  $\Delta$  is conveniently parameterized by

$$\Delta = \cos \gamma, \quad 0 \leq \gamma < \pi. \quad (2.2)$$

We will give our analytical results for the entire regime  $0 \leq \gamma < \pi$  ( $-1 < \Delta \leq 1$ ), numerical results will be given for the repulsive range  $0 \leq \gamma \leq \pi/2$  ( $0 \leq \Delta \leq 1$ ).

Our first goal is the determination of the thermal current operator  $j^E$ . To this end we impose the continuity equation relating the time derivative of the local Hamiltonian (interaction) and the divergence of the current:  $\dot{h} = -\text{div } j^E$ . The time evolution of the local Hamiltonian in (2.1) is obtained from the commutator with the total Hamiltonian and the divergence of the local current on the lattice is given by a difference expression

$$\frac{\partial h_{kk+1}(t)}{\partial t} = i[\mathcal{H}, h_{kk+1}(t)] = -\{j_{k+1}^E(t) - j_k^E(t)\}. \quad (2.3)$$

Apparently the last equation is satisfied with a local thermal current operator  $j_k^E$  defined by

$$j_k^E = i[h_{k-1k}, h_{kk+1}]. \quad (2.4)$$

In fact the total thermal current  $\mathcal{J}_E = \sum_{k=1}^L j_k^E$

$$\mathcal{J}_E = -iJ^2 \sum_{k=1}^L \left\{ \sigma_k^z (\sigma_{k-1}^+ \sigma_{k+1}^- - \sigma_{k+1}^+ \sigma_{k-1}^-) - \Delta (\sigma_{k-1}^z + \sigma_{k+2}^z) (\sigma_k^+ \sigma_{k+1}^- - \sigma_{k+1}^+ \sigma_k^-) \right\}, \quad (2.5)$$

commutes with the Hamiltonian,  $[\mathcal{H}, \mathcal{J}_E] = 0$ , as it is closely connected with a nontrivial conserved quantity derived from the underlying integrability of the model. To see this, we introduce the transfer matrix constructed from the  $R$ -matrix  $R \in \text{End}(V \otimes V)$  where  $V$  denotes a two dimensional irreducible module over the quantum algebra  $U_q(\widehat{\mathfrak{sl}}(2))$ . The nonzero 6 elements of the  $R$ -matrix with the spectral parameter  $v$  are given by

$$R_{11}^{11}(v) = R_{22}^{22}(v) = \frac{[v+2]}{[2]}, \quad R_{12}^{12}(v) = R_{21}^{21}(v) = \frac{[v]}{[2]}, \quad R_{12}^{21}(v) = R_{21}^{12}(v) = 1, \quad (2.6)$$

where  $[v]$  denotes  $[v] = \sin(\gamma v/2)/\sin(\gamma/2)$  and the indices of the above relations can be interpreted as

$$\begin{aligned} |1\rangle &= |\uparrow\rangle, & |2\rangle &= |\downarrow\rangle, \\ R(v)|\alpha\rangle \otimes |\beta\rangle &= \sum_{\gamma,\delta} |\gamma\rangle \otimes |\delta\rangle R_{\alpha\beta}^{\gamma\delta}(v). \end{aligned} \quad (2.7)$$

The  $R$ -matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \quad (2.8)$$

Due to the YBE the transfer matrix  $T(v) = \widehat{\prod}_k R_{kk+1}(v)$  is commutative with respect to different spectral parameter  $v$  and  $v'$ , i.e

$$[T(v), T(v')] = 0. \quad (2.9)$$

As is well known,  $\ln T(v)$  is the generating function for the conserved quantities

$$\mathcal{J}^{(n)} = \left( \frac{\partial}{\partial v} \right)^n \ln T(v) \Big|_{v=0}. \quad (2.10)$$

In particular the Hamiltonian (2.1) and the thermal current (2.5) are expressed in terms of  $\mathcal{J}^{(1)}$  and  $\mathcal{J}^{(2)}$ , respectively. Explicitly they read

$$\begin{aligned} \mathcal{H} &= \frac{2J \sin \gamma}{\gamma} \mathcal{J}^{(1)} - \frac{JL}{2} \Delta, \\ \mathcal{J}_E &= i \left( \frac{2J \sin \gamma}{\gamma} \right)^2 \mathcal{J}^{(2)} + iJ^2 L. \end{aligned} \quad (2.11)$$

Due to the commutativity (2.9), every operator  $\mathcal{J}^{(n)}$  commutes with the Hamiltonian  $\mathcal{H}$ .

### 3 Thermal conductivity

Our goal is the calculation of the second moment of the thermal current. Quite generally the expectation values of conserved quantities may be calculated by use of a suitable generating function. As such we want to define a modified partition function

$$Z = \text{Tr} \exp(-\beta \mathcal{H} + \lambda \mathcal{J}_E), \quad (3.1)$$

from which we find the expectation values by derivatives with respect to  $\lambda$  at  $\lambda = 0$

$$\frac{\partial}{\partial \lambda} \ln Z \Big|_{\lambda=0} = \langle \mathcal{J}_E \rangle = 0, \quad \left( \frac{\partial}{\partial \lambda} \right)^2 \ln Z \Big|_{\lambda=0} = \langle \mathcal{J}_E^2 \rangle - \langle \mathcal{J}_E \rangle^2 = \langle \mathcal{J}_E^2 \rangle, \quad (3.2)$$

where we have used that the expectation value of the thermal current in thermodynamical equilibrium is zero.

Instead of  $Z$  we will find it slightly more convenient to work with a partition function

$$\overline{Z} = \text{Tr} \exp(-\lambda_1 \mathcal{J}^{(1)} - \lambda_n \mathcal{J}^{(n)}). \quad (3.3)$$

With view to (2.11) we choose

$$\lambda_1 = \beta \frac{2J \sin \gamma}{\gamma}, \quad \lambda_2 = -i\lambda \left( \frac{2J \sin \gamma}{\gamma} \right)^2, \quad (3.4)$$

and obtain the desired expectation values from  $\overline{Z}$

$$\langle \mathcal{J}_E^2 \rangle = \left( \frac{\partial}{\partial \lambda} \right)^2 \ln \overline{Z} \Big|_{\lambda=0}. \quad (3.5)$$

We can deal with  $\overline{Z}$  rather easily. Consider the trace of a product of  $N$  row-to-row transfer matrices  $T(u_j)$  with some spectral parameters  $u_j$  close to zero, but still to be specified, and the  $N$ th power of the inverse of  $T(0)$

$$\begin{aligned} Z_N &= \text{Tr} [T(u_1) \cdot \dots \cdot T(u_N) \cdot T(0)^{-N}] \\ &= \text{Tr} \exp \left( \sum_j [\ln T(u_j) - \ln T(0)] \right). \end{aligned}$$

Now it is a standard exercise in arithmetic to devise a sequence of  $N$  numbers  $u_1, \dots, u_N$  (actually  $u_j = u_j^{(N)}$ ) such that

$$\lim_{N \rightarrow \infty} \sum_j [f(u_j) - f(0)] = -\lambda_1 \frac{\partial}{\partial v} f(v) \Big|_{v=0} - \lambda_n \left( \frac{\partial}{\partial v} \right)^n f(v) \Big|_{v=0}. \quad (3.7)$$

We only need the existence of such a sequence of numbers, the precise values are of no importance. In the limit  $N \rightarrow \infty$  we note

$$\lim_{N \rightarrow \infty} Z_N = \overline{Z}. \quad (3.8)$$

We can proceed along the established path of the quantum transfer matrix (QTM) formalism [16–22] and derive the partition function  $Z_N$  in the thermodynamic limit  $L \rightarrow \infty$

$$\lim_{L \rightarrow \infty} Z_N^{1/L} = \Lambda, \quad (3.9)$$

where  $\Lambda$  is the largest eigenvalue of the QTM. The integral expression for  $\Lambda$  reads

$$\ln \Lambda = \sum_j [e(u_j) - e(0)] + \int_{-\infty}^{\infty} K(v) \ln[\mathfrak{A}(v) \overline{\mathfrak{A}}(v)] dv, \quad K(v) = \frac{1}{4 \cosh \frac{\pi}{2} v}, \quad (3.10)$$

with some function  $e(v)$  given in [21]. In the limit  $N \rightarrow \infty$  the first term on the r.h.s. of the last equation turns into

$$\lim_{N \rightarrow \infty} \sum_j [e(u_j) - e(0)] = -\lambda_1 \frac{\partial}{\partial v} e(v) \Big|_{v=0} - \lambda_n \left( \frac{\partial}{\partial v} \right)^n e(v) \Big|_{v=0}, \quad (3.11)$$

a rather irrelevant term as it is linear in  $\lambda_1$  and  $\lambda_n$ , and therefore the second derivatives with respect to  $\lambda_1$  and  $\lambda_n$  vanish. The functions  $\mathfrak{A}(v)$  and  $\overline{\mathfrak{A}}(v)$  are determined from the following set of non-linear integral equations (NLIEs):

$$\begin{aligned} \ln \mathfrak{a}(v) &= \sum_j [\varepsilon_0(v - iu_j) - \varepsilon_0(0)] + \kappa * \ln \mathfrak{A}(v) - \kappa * \ln \overline{\mathfrak{A}}(v + 2i), \\ \ln \overline{\mathfrak{a}}(v) &= \sum_j [\varepsilon_0(v - iu_j) - \varepsilon_0(0)] + \kappa * \ln \overline{\mathfrak{A}}(v) - \kappa * \ln \mathfrak{A}(v - 2i), \\ \mathfrak{A}(v) &= 1 + \mathfrak{a}(v), \quad \overline{\mathfrak{A}}(v) = 1 + \overline{\mathfrak{a}}(v). \end{aligned} \quad (3.12)$$

with a function  $\varepsilon_0(v)$  given in terms of hyperbolic functions [21]. The symbol  $*$  denotes the convolution  $f * g(v) = \int_{-\infty}^{\infty} f(v - v')g(v')dv$  and the function  $\kappa(v)$  is defined by

$$\kappa(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sinh\left(\frac{\pi}{\gamma} - 2\right)k}{2 \cosh k \sinh\left(\frac{\pi}{\gamma} - 1\right)k} e^{ikv} dk. \quad (3.13)$$

Again, the summations in (3.12) can be simplified in the limit  $N \rightarrow \infty$

$$\lim_{N \rightarrow \infty} \sum_j [\varepsilon_0(v - iu_j) - \varepsilon_0(v)] = -\lambda_1 \underbrace{\left(-i \frac{\partial}{\partial v}\right) \varepsilon_0(v)}_{=:\varepsilon_1(v)} - \lambda_n \underbrace{\left(-i \frac{\partial}{\partial v}\right)^n \varepsilon_0(v)}_{=:\varepsilon_n(v)}. \quad (3.14)$$

where the first function can be found in [21] and is simply

$$\varepsilon_1(v) = 2\pi K(v) = \frac{\pi}{2 \cosh \frac{\pi}{2}v}, \quad (3.15)$$

and hence the second function is

$$\varepsilon_n(v) = \left(-i \frac{\partial}{\partial v}\right)^{n-1} \varepsilon_1(v). \quad (3.16)$$

We like to note that the structure of the driving term (3.14) appearing in the NLIE (3.12) reflects the structure of the generalized Hamiltonian in the exponent on the r.h.s. of (3.3). We could have given an alternative derivation of the NLIE along the lines of the thermodynamic Bethe Ansatz (TBA). In such an approach the driving term is typically the one-particle energy corresponding to the generalized Hamiltonian. Hence it has contributions due to the first as well as the  $n$ th logarithmic derivative of the row-to-row transfer matrix, i.e. the terms  $\varepsilon_1$  and  $\varepsilon_n$ .

In Fig.1 (a) we show  $\tilde{\kappa}(T)$  for various anisotropy parameters  $\gamma$ . Note that  $\tilde{\kappa}(T)$  has linear  $T$  dependence at low temperatures. In the free fermion case ( $\gamma = \pi/2$ ) we observe that the ratio of the thermal conductivity  $\kappa$  to the electrical conductivity  $\sigma$  obeys the Wiedemann-Franz law (see the next section). There is a finite temperature maximum at roughly half of the temperature of the maximum in the specific heat  $c(T)$  data. At high temperatures  $\tilde{\kappa}(T)$  behaves like  $1/T^2$ . This and the low temperature asymptotics will be studied analytically in the next section.

In general the data of  $\tilde{\kappa}(T)$  show a much stronger variation with the anisotropy  $\gamma$  than the specific heat  $c(T)$  data. In Fig.1 (b) we show the ratio  $\tilde{\kappa}(T)/c(T)$  which have finite low and high temperature limits and strongly depend on the anisotropy parameter.

## 4 Low and high temperature limits

In certain limits we can analytically evaluate the NLIE. For high temperatures the NLIE linearize and hence can be solved. At low temperatures the NLIE remain non-linear, but the driving terms simplify. Then the symmetry of the integration kernel allows for an evaluation of the physically interesting quantities by avoiding the explicit solution of the NLIE.

### 4.1 Low temperature limit

We consider the low-temperature behavior of the current-current correlation function of the thermal current  $\mathcal{J}_E$  by making use of the dilogarithm trick. The functions  $\mathfrak{a}(v)$  and  $\bar{\mathfrak{a}}(v)$  in the NLIEs (3.12) exhibit a crossover behavior

$$\begin{aligned} \mathfrak{a}(v) &\ll 1, \quad \bar{\mathfrak{a}}(v) \ll 1 \quad \text{for} \quad -\mathcal{K}_- < v < \mathcal{K}_+, \\ \mathfrak{a}(v) &\simeq 1, \quad \bar{\mathfrak{a}}(v) \simeq 1 \quad \text{for} \quad v < -\mathcal{K}_-, \mathcal{K}_+ < v, \end{aligned} \quad (4.1)$$

where

$$\mathcal{K}_{\pm} = \frac{2}{\pi} \ln \left\{ \frac{2\pi J \sin \gamma}{\gamma} \left( \beta \pm \frac{\lambda \pi J \sin \gamma}{\gamma} \right) \right\}. \quad (4.2)$$

We introduce the scaling functions

$$a_{\pm}(v) = \mathfrak{a} \left( \pm \frac{2}{\pi} v \pm \mathcal{K}_{\pm} \right), \quad \bar{a}_{\pm}(v) = \bar{\mathfrak{a}} \left( \pm \frac{2}{\pi} v \pm \mathcal{K}_{\pm} \right), \quad (4.3)$$

and similarly the functions  $A_{\pm}$  ( $\bar{A}_{\pm}$ ) for  $\mathfrak{A}$  ( $\bar{\mathfrak{A}}$ ). From the NLIEs (3.12), one sees the scaling functions satisfy the following “scaled” equations in the limit  $\beta \rightarrow \infty$ :

$$\begin{aligned} \ln a_{\pm}(v) &= -e^{-v} + \kappa_1 * \ln A_{\pm}(v) - \kappa_{2\pm} * \ln \bar{A}_{\pm}(v), \\ \ln \bar{a}_{\pm}(v) &= -e^{-v} + \kappa_1 * \ln \bar{A}_{\pm}(v) - \kappa_{2\mp} * \ln A_{\pm}(v), \\ \kappa_1(v) &= \frac{2}{\pi} \kappa \left( \frac{2v}{\pi} \right), \quad \kappa_{2\pm}(v) = \frac{2}{\pi} \kappa \left( \frac{2v}{\pi} \pm 2i \right). \end{aligned} \quad (4.4)$$

In this limit, the integrals of the functions  $\mathfrak{A}$ ,  $\overline{\mathfrak{A}}$  in eq. (3.10) can be written as

$$\begin{aligned}\ln \Lambda &= \int_{-\infty}^{\infty} K(v) \ln[\mathfrak{A}(v)\overline{\mathfrak{A}}(v)] dv \\ &= \frac{\gamma^2}{2\pi^2 J \sin \gamma (\beta\gamma + \lambda\pi J \sin \gamma)} \int_{-\infty}^{\infty} e^{-v} (\ln A_+(v) + \ln \overline{A}_+(v)) dv \\ &\quad + \frac{\gamma^2}{2\pi^2 J \sin \gamma (\beta\gamma - \lambda\pi J \sin \gamma)} \int_{-\infty}^{\infty} e^{-v} (\ln A_-(v) + \ln \overline{A}_-(v)) dv.\end{aligned}\quad (4.5)$$

The right hand side of the above equation is evaluated as follows. (A) After taking the derivative of the first and second equation in (4.4), we multiply them by  $\ln A_{\pm}(v)$  and  $\ln \overline{A}_{\pm}(v)$ , respectively and take the summation of them with respect to each side. (B) Next we multiply (4.4) by  $[\ln A_{\pm}(v)]'$  and  $[\ln \overline{A}_{\pm}(v)]'$ , respectively and take the summation. Next we subtract the resultant equation of (B) from the one of (A). After integrating over  $v$ , we obtain

$$D_{\pm} = 2 \int_{-\infty}^{\infty} e^{-v} [\ln A_{\pm}(v) + \ln \overline{A}_{\pm}(v)] dv, \quad (4.6)$$

where

$$\begin{aligned}D_{\pm} &= \int_{-\infty}^{\infty} \left( \ln A_{\pm}(v) \frac{d}{dv} \ln a_{\pm}(v) + \ln \overline{A}_{\pm}(v) \frac{d}{dx} \ln \overline{a}_{\pm}(v) \right. \\ &\quad \left. - \ln a_{\pm}(v) \frac{d}{dv} \ln A_{\pm}(v) - \ln \overline{a}_{\pm}(v) \frac{d}{dx} \ln \overline{A}_{\pm}(v) \right) dv \\ &= \int_{a_{\pm}(-\infty)}^{a_{\pm}(\infty)} \left( \frac{\ln(1+a)}{a} - \frac{\ln a}{1+a} \right) da + \int_{\overline{a}_{\pm}(-\infty)}^{\overline{a}_{\pm}(\infty)} \left( \frac{\ln(1+\overline{a})}{\overline{a}} - \frac{\ln \overline{a}}{1+\overline{a}} \right) d\overline{a}.\end{aligned}\quad (4.7)$$

The quantities  $D_{\pm}$  can be expressed in terms of Roger's dilogarithm  $\mathcal{L}(v)$

$$\begin{aligned}D_{\pm} &= 2\mathcal{L}\left(\frac{a_{\pm}(\infty)}{1+a_{\pm}(\infty)}\right) + 2\mathcal{L}\left(\frac{\overline{a}_{\pm}(\infty)}{1+\overline{a}_{\pm}(\infty)}\right) \\ &\quad - 2\mathcal{L}\left(\frac{a_{\pm}(-\infty)}{1+a_{\pm}(-\infty)}\right) - 2\mathcal{L}\left(\frac{\overline{a}_{\pm}(-\infty)}{1+\overline{a}_{\pm}(-\infty)}\right), \\ \mathcal{L}(v) &= -\frac{1}{2} \int_0^v \left( \frac{\ln(1-x)}{x} + \frac{\ln x}{1-x} \right) dx.\end{aligned}\quad (4.8)$$

Using the asymptotic value of the scaling functions  $a_{\pm}(\pm\infty) = \overline{a}_{\pm}(\pm\infty) = 1$  and substituting eq. (4.6) for eq. (4.5), we arrive at

$$\ln \Lambda = \frac{\gamma^2}{12\beta J \sin \gamma} \left( \frac{1}{\beta\gamma + \lambda\pi J \sin \gamma} + \frac{1}{\beta\gamma - \lambda\pi J \sin \gamma} \right). \quad (4.9)$$

Here we have used the identity  $\mathcal{L}(v) + \mathcal{L}(1-v) = \pi^2/6$ . Hence the low-temperature asymptotics of the current-current correlation function is evaluated to

$$\langle \mathcal{J}_{\text{E}}^2 \rangle \simeq \frac{J\pi^2 \sin \gamma}{3\beta^3 \gamma} + O\left(\frac{1}{\beta^4}\right). \quad (4.10)$$



From this result we see that  $\tilde{\kappa}(T)$  is linear in  $T$  at low  $T$

$$\tilde{\kappa}(T) \simeq \frac{\pi^2}{3} v T, \quad (4.11)$$

with  $v = J\pi \sin \gamma / \gamma$  the velocity of the elementary excitations. From the low temperature behavior of the specific heat  $c(T) = (\pi/3v)T$  we find

$$\frac{\tilde{\kappa}(T)}{c(T)} \rightarrow \pi v^2. \quad (4.12)$$

Finally we want to compare the thermal conductivity  $\kappa$  and the electrical conductivity  $\sigma$  which in the low temperature limit can be described by the Drude weight  $D_c$  [24]

$$\text{Re } \sigma(\omega) = 2\pi D_c \delta(\omega), \quad D_c = \frac{v}{4(\pi - \gamma)}. \quad (4.13)$$

This yields

$$\frac{\kappa}{\sigma} \simeq \frac{2}{3} \pi (\pi - \gamma) T, \quad (T \rightarrow 0), \quad (4.14)$$

which in the free fermion case ( $\gamma = \pi/2$ ) gives the Wiedemann-Franz law.

## 4.2 High temperature limit

In the high temperature limit ( $\beta \rightarrow 0$ ), the auxiliary functions satisfy the following integral equations linear in  $(\partial/\partial\lambda)^2 \ln \mathfrak{A}$

$$\begin{aligned} \left(\frac{\partial}{\partial\lambda}\right)^2 \ln \mathfrak{A}(v) &= \left(\frac{\partial}{\partial\lambda} \ln \mathfrak{A}(v)\right)^2 + \frac{1}{2} \kappa * \left(\frac{\partial}{\partial\lambda}\right)^2 \ln \mathfrak{A}(v) - \frac{1}{2} \kappa * \left(\frac{\partial}{\partial\lambda}\right)^2 \ln \overline{\mathfrak{A}}(v + 2i), \\ \left(\frac{\partial}{\partial\lambda}\right)^2 \ln \overline{\mathfrak{A}}(v) &= \left(\frac{\partial}{\partial\lambda} \ln \overline{\mathfrak{A}}(v)\right)^2 + \frac{1}{2} \kappa * \left(\frac{\partial}{\partial\lambda}\right)^2 \ln \overline{\mathfrak{A}}(v) - \frac{1}{2} \kappa * \left(\frac{\partial}{\partial\lambda}\right)^2 \ln \mathfrak{A}(v - 2i). \end{aligned} \quad (4.15)$$

Here we have used the high temperature asymptotics  $\mathfrak{a}(v), \overline{\mathfrak{a}}(v) \simeq 1$ . By use of the dressed function formalism, we obtain the identity

$$\begin{aligned} \left(\frac{\partial}{\partial\lambda}\right)^2 \ln \Lambda \Big|_{\lambda=0} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \varepsilon_1(v) \left(\frac{\partial}{\partial\lambda}\right)^2 \ln \mathfrak{A}(v) \overline{\mathfrak{A}}(v) \Big|_{\lambda=0} dv \\ &= -\frac{\gamma}{8\pi J \sin \gamma} \int_{-\infty}^{\infty} \frac{\partial \mathfrak{a}(v)}{\partial \beta} \left(\frac{\partial \mathfrak{a}(v)}{\partial \lambda}\right)^2 \Big|_{\lambda=0} dv. \end{aligned} \quad (4.16)$$

The integrand in the above equation is found analytically

$$\begin{aligned} \frac{\partial \mathfrak{a}(v)}{\partial \beta} \Big|_{\lambda=0} &= \frac{J \sin^2 \gamma}{2 \sinh \frac{\gamma}{2}(v + i)} \left( \frac{1}{\sinh \frac{\gamma}{2}(v + 3i)} - \frac{1}{\sinh \frac{\gamma}{2}(v - i)} \right), \\ \frac{\partial \mathfrak{a}(v)}{\partial \lambda} \Big|_{\lambda=0} &= -\frac{\partial}{\partial v} \left( \frac{\partial \mathfrak{a}(v)}{\partial \beta} \right) \Big|_{\lambda=0}. \end{aligned} \quad (4.17)$$

By use of these explicit expressions we obtain the high temperature limit of  $\tilde{\kappa}(T)$

$$\tilde{\kappa}(T) \simeq \frac{J^4}{4} \left( 3 + \frac{\sin 3\gamma}{\sin \gamma} \right) \beta^2 + O(\beta^3). \quad (4.18)$$

This result generalizes the known results for the special cases  $\gamma = \pi/2$  (free fermion model) in [25] and  $\gamma = 0$  (isotropic Heisenberg chain) in [26].

## 5 Summary and discussion

In this work we have presented a method for the calculation of the thermal conductivity  $\kappa(\omega)$  of the integrable spin-1/2  $XXZ$  chain. The investigation of this system is drastically simplified in comparison to other systems as here the thermal current is a conserved quantity. This led to  $\text{Re } \kappa(\omega) = \tilde{\kappa}\delta(\omega)$  where the weight  $\tilde{\kappa}$  was calculated for arbitrary temperature and various anisotropy parameters in the repulsive, critical regime of the  $XXZ$  model.

We like to comment on possible generalizations of our investigation. First of all, the computation of  $\kappa(q, \omega)$ , i.e. the thermal conductivity for a thermal current operator  $\mathcal{J}_E(q)$  with non-zero momentum  $q$ , would be desirable. Unfortunately, it is only the case  $q = 0$  that allows for an analytical approach. On the other hand, for small values of  $q$  a behavior of  $\kappa(q, \omega)$  similar to the  $q = 0$  case is expected with the  $\delta(\omega)$  factor to be replaced by  $\delta(\omega - vq)$  where  $v$  is the velocity of the elementary excitations. Finally, the investigation of the thermal conductivity of the general  $XXZ$  chain with arbitrary anisotropies in the easy plane (gapless) and easy axis (gapped) regimes should be possible. This will be reported elsewhere.

## Acknowledgments

We would like to thank B. Büchner, C. Gros, D. Rainer, and X. Zotos for stimulating discussions. The authors acknowledge financial support by the Deutsche Forschungsgemeinschaft under grants No. Kl 645/3-3, 645/4-1 and the Schwerpunktprogramm SP1073.

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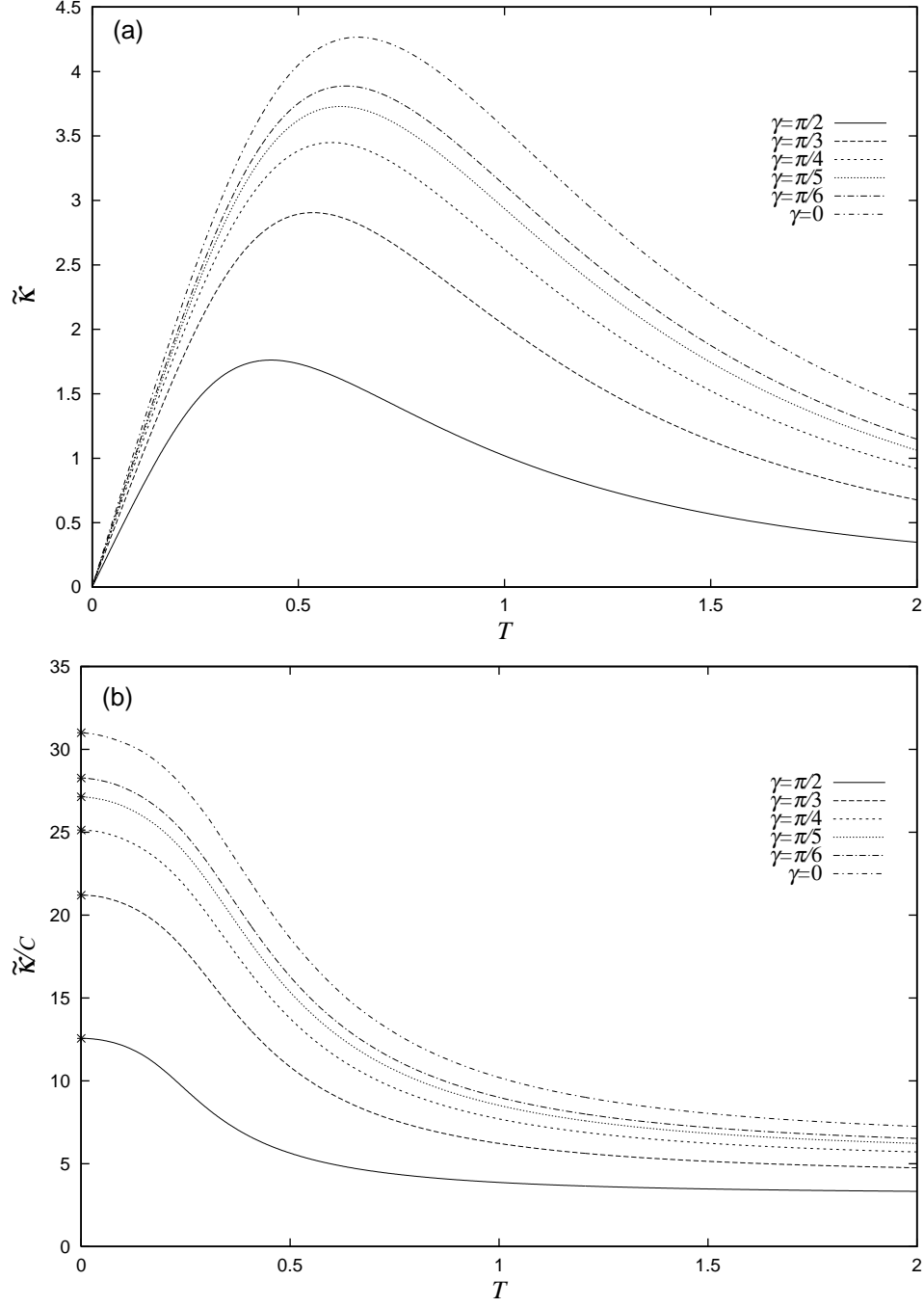


Figure 1: (a) Illustration of numerical results for the thermal conductivity  $\tilde{\kappa}$  as a function of temperature  $T$  for various anisotropy parameters  $\gamma = 0, \pi/6, \pi/5, \pi/4, \pi/3, \pi/2$ . (b) Depiction of the ratio of thermal conductivity and specific heat  $\tilde{\kappa}/c$  as a function of temperature. The analytic result (4.12) in the low temperature limit is also depicted by the symbol \*.